Poisson Brackets

$$\{f,g\} = \sum_k \left(\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial p_k} \right)$$

Properties of the Brackets will be proven in HW.

- $\{f,g\} = -\{g,f\}$
- $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$ for $\alpha,\beta \in \mathbb{R}$
- $\{fg,h\} = f\{g,h\} + \{f,h\}g = -\{h,fg\}$
- Jacobi $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

We also have

- $\{q_i,q_j\} = 0$, $\{p_i,p_j\} = 0$, $\{q_i,p_j\} = \delta_{ij}$ $\leftarrow$ Look familiar?

- $\dfrac{df}{dt} = \{f,H\} + \dfrac{\partial f}{\partial t}$ – Hamiltonian generates time evolution.

- $\dfrac{dH}{dt} = \underbrace{\{H,H\}}_{=0\text{, trivially}} + \dfrac{\partial H}{\partial t}$ $\Rightarrow$

  $\{f,H\} = 0 \Rightarrow \dfrac{df}{dt} = \dfrac{\partial f}{\partial t}$

  $f(p,q)$ with $\{f,H\} = 0 \Rightarrow df/dt = 0$.

  $H = \text{const.}$ if $\underline{\dfrac{\partial H}{\partial t} = 0}$.

Example: Angular Momentum

$$L_i = \sum_{jk} \varepsilon_{ijk} x_j p_k$$

$$\Rightarrow \{L_i, L_r\} = \sum_{jk}\sum_{st} \varepsilon_{ijk}\varepsilon_{rst}\{x_j p_k, x_s p_t\}$$

$$= \sum_{jk}\sum_{st} \varepsilon_{ijk}\varepsilon_{rst}\left(x_j\{p_k, x_s p_t\} + \{x_j, x_s p_t\} p_k \right)$$

$$= \sum_{jk}\sum_{st} \varepsilon_{ijk}\varepsilon_{rst}\Big(x_j\big(-x_s \overbrace{\{p_t,p_k\}}^{=0} - \{x_s,p_k\}p_t\big) + \big(-x_s\{p_t,x_j\} - \underbrace{\{x_s,x_j\}}_{=0}p_t\big)p_k \Big)$$

$$= \sum_{jk}\sum_{st} \varepsilon_{ijk}\varepsilon_{rst}\left(-x_j p_t \delta_{ks} + x_s \delta_{jt} p_k \right)$$

$$= \sum_{jk}\sum_{st} \varepsilon_{ijk}\varepsilon_{rst}\left(x_s p_k \delta_{jt} - x_j p_t \delta_{ks} \right)$$

$$= \sum_{jks} \varepsilon_{ijk}\varepsilon_{rsj} x_s p_k - \sum_{jkt} \varepsilon_{ijk}\varepsilon_{rkt} x_j p_t$$

$$= \sum_{jks} \varepsilon_{kij}\varepsilon_{rsj} x_s p_k - \sum_{jkt} \varepsilon_{ijk}\varepsilon_{trk} x_j p_t$$

$$= \sum_{ks} \left(\delta_{kr}\delta_{is} - \delta_{ks}\delta_{ir}\right) x_s p_k - \sum_{jt}\left(\delta_{it}\delta_{jr} - \delta_{ir}\delta_{jt}\right) x_j p_t$$

$$= x_i p_r - \sum_s (x_s p_s)\,\delta_{ir} - x_r p_i + \sum_t (x_t p_t)\,\delta_{ir} = x_i p_r - x_r p_i$$

Make the example explicit.

$$L_1 = x_2 p_3 - x_3 p_2 \qquad \text{etc.}$$

$$\{L_1, L_2\} = \{x_2 p_3 - x_3 p_2,\ x_3 p_1 - x_1 p_3\} = \{x_2 p_3, x_3 p_1\} - \{x_2 p_3, x_1 p_3\}$$
$$- \{x_3 p_2, x_3 p_1\} + \{x_3 p_2, x_1 p_3\}$$

$$\cancel{\{AB, CD\} = A\{B, CD\} + \{A, CD\}B = -A\{CD, B\} - \{CD, A\}B}$$
$$\cancel{= -AC\{D, B\} - A\{C, B\}D - C\{D, A\}B - \{D, A\}CB}$$
$$\cancel{\Rightarrow \{x_2 p_3, x_3 p_1\} = -x_2 x_3 \underbrace{\{p_2, p_3\}}_{=0} - p_1 x_2}$$

$$\{AB, CD\} = A\{B, CD\} + \{A, CD\}B = -A\{CD, B\} - \{CD, A\}B$$
$$= -AC\{D, B\} - A\{C, B\}D - C\{D, A\}B - \{C, A\}DB$$
$$= AC\{B, D\} + A\{B, C\}D + C\{A, D\}B + \{A, C\}DB$$

(if we have to worry about non-commutativity... of course, we don't have to here).

↝ We will only get contributions if we have x_i and p_i in the same bracket, so

$$\{x_2 p_3, x_3 p_1\} = x_2 x_3 \cancel{\{p_3, p_1\}}^{\to 0} + x_2 \underbrace{\{p_3, x_3\}}_{=-1} p_1 + x_3 \cancel{\{x_2, p_1\}}^{\to 0} p_3 + \cancel{\{x_2, x_3\}}^{\to 0} p_1 p_3$$
$$= -x_2 p_1$$

$$\{x_2 p_3, x_1 p_3\} = 0$$

$$\{x_3 p_2, x_3 p_1\} = 0$$

$$\{x_3 p_2, x_1 p_3\} = x_1 x_3 \cancel{\{p_2, p_3\}}^{\to 0} + x_3 \cancel{\{p_2, x_1\}}^{\to 0} p_3 + x_1 \underbrace{\{x_3, p_3\}}_{=1} p_2 + \cancel{\{x_3, x_1\}}^{\to 0} p_3 p_2$$
$$= x_1 p_2$$

$$\Rightarrow \{L_1, L_2\} = x_1 p_2 - x_2 p_1 = L_3 \qquad \text{etc.}$$

⟹ Lenz - Runge as ungraded HW or practice problem.

Canonical Transformations

- Lagrange equations are invariant under general coordinate trafos in configuration space
- Hamiltonian formalism treats q_j, p_j as independent, so let's look for form invariance in phase space

↳ will make it possible to find canonical variables that simplify H and make it easy to solve Hamilton's equs.:

$$q_j \to Q_j(q_k, p_k) \qquad p_j \to P_j(q_k, p_k)$$

We can write Hamilton's equs. in symplectic form (symplectic, gr., "intertwined").

$$\vec{x} = (q_1, \dots, q_n, p_1, \dots, p_n)^T \quad ; \quad \frac{\partial H}{\partial \vec{x}} = \left(\frac{\partial H}{\partial q_1}, \dots, \frac{\partial H}{\partial q_n}, \frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n}\right)^T$$

$$J = \begin{array}{cc} q_j & p_j \end{array} \atop \left(\begin{array}{c|c} & \mathbb{1} \\ \hline -\mathbb{1} & \end{array}\right)$$

$$\Rightarrow \boxed{\dot{\vec{x}} = J \frac{\partial H}{\partial \vec{x}}}$$

Work in symplectic form and make the transformation

$$x_j \to y_j(x_k)$$

$$\dot{y}_j = \sum_{k=1}^{2n} \frac{\partial y_j}{\partial x_k} \dot{x}_k = \sum_{kl} \frac{\partial y_j}{\partial x_k} J_{kl} \frac{\partial H}{\partial x_l} = \sum_{klm} \frac{\partial y_j}{\partial x_k} J_{kl} \frac{\partial H}{\partial y_m} \frac{\partial y_m}{\partial x_l}$$

$$= \sum_{klm} \frac{\partial y_j}{\partial x_k} J_{kl} \frac{\partial y_m}{\partial x_l} \frac{\partial H}{\partial y_m}$$

Define the Jacobian of the coordinate change

$$M_{jk} = \frac{\partial y_j}{\partial x_k}$$

$$\Rightarrow \dot{y}_j = \sum_{klm} M_{jk} J_{kl} M^T_{lm} \frac{\partial H}{\partial y_m} \Rightarrow \dot{\vec{y}} = (M J M^T) \frac{\partial H}{\partial \vec{y}}$$

Thus, Hamilton's equs. are form invariant of

$$M J M^T = J \iff \sum_{lm} \frac{\partial y_j}{\partial x_l} J_{lm} \frac{\partial y_m}{\partial x_k} = J_{jk}$$

If this relationship holds, the Jacobian is called symplectic and the transformation $x_j \to y_j(x_k)$ is canonical.

The Poisson brackets are invariant under canonical transformations. Conversely, a trafo which preserves the fundamental Poisson brackets ~~~~

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0 \;, \quad \{Q_i, P_j\} = \delta_{ij}$$

is canonical.

Proof
$$\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right) = \sum_{\underset{1\dots 2n}{rs}} \frac{\partial f}{\partial x_r} J_{rs} \frac{\partial g}{\partial x_s}$$

Now transform: $\displaystyle \frac{\partial f}{\partial x_r} = \sum_s \frac{\partial f}{\partial y_s} \frac{\partial y_s}{\partial x_r} = \sum_s \frac{\partial f}{\partial y_s} M_{sr}$

$$\hookrightarrow \quad \{f, g\}_x = \sum_{rstu} \frac{\partial f}{\partial y_t} M_{tr} J_{rs} M^T_{su} \frac{\partial g}{\partial y_u} = \sum_{tu} \frac{\partial f}{\partial y_t} (M J M^T)_{tu} \frac{\partial g}{\partial y_u}$$

$$= \sum_{tu} \frac{\partial f}{\partial y_t} J_{tu} \frac{\partial g}{\partial y_u} = \{f, g\}_y$$

We now consider the other direction: Use the "old" notation

$$\{f, g\}_{(q,p)} = \sum_j \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q_j} \frac{\partial f}{\partial p_j} \right)$$

$$= \sum_{jkl} \left[\left(\frac{\partial f}{\partial Q_k} \frac{\partial Q_k}{\partial q_j} + \frac{\partial f}{\partial P_k} \frac{\partial P_k}{\partial q_j} \right) \left(\frac{\partial g}{\partial Q_l} \frac{\partial Q_l}{\partial p_j} + \frac{\partial g}{\partial P_l} \frac{\partial P_l}{\partial p_j} \right) - \left(\frac{\partial g}{\partial Q_l} \frac{\partial Q_l}{\partial q_j} + \frac{\partial g}{\partial P_l} \frac{\partial P_l}{\partial q_j} \right) \left(\frac{\partial f}{\partial Q_k} \frac{\partial Q_k}{\partial p_j} + \frac{\partial f}{\partial P_k} \frac{\partial P_k}{\partial p_j} \right) \right]$$

$$= \sum_{jkl} \left[\frac{\partial f}{\partial Q_k} \frac{\partial g}{\partial P_l} \left(\frac{\partial Q_k}{\partial q_j} \frac{\partial P_l}{\partial p_j} - \frac{\partial P_l}{\partial q_j} \frac{\partial Q_k}{\partial p_j} \right) + \frac{\partial f}{\partial P_k} \frac{\partial g}{\partial Q_l} \left(\frac{\partial P_k}{\partial q_j} \frac{\partial Q_l}{\partial p_j} - \frac{\partial Q_l}{\partial q_j} \frac{\partial P_k}{\partial p_j} \right) \right.$$
$$\left. + \frac{\partial f}{\partial Q_k} \frac{\partial g}{\partial Q_l} \left(\frac{\partial Q_k}{\partial q_j} \frac{\partial Q_l}{\partial p_j} - \frac{\partial Q_l}{\partial q_j} \frac{\partial Q_k}{\partial p_j} \right) + \frac{\partial f}{\partial P_k} \frac{\partial g}{\partial P_l} \left(\frac{\partial P_k}{\partial q_j} \frac{\partial P_l}{\partial p_j} - \frac{\partial P_l}{\partial q_j} \frac{\partial P_k}{\partial p_j} \right) \right]$$

$$= \sum_{kl} \left[\frac{\partial f}{\partial Q_k} \frac{\partial g}{\partial P_l} \{Q_k, P_l\}_{(q,p)} + \frac{\partial f}{\partial P_k} \frac{\partial g}{\partial Q_l} \{P_k, Q_l\}_{(q,p)} \right.$$
$$\left. + \frac{\partial f}{\partial Q_k} \frac{\partial g}{\partial Q_l} \{Q_k, Q_l\}_{(q,p)} + \frac{\partial f}{\partial P_k} \frac{\partial g}{\partial P_l} \{P_k, P_l\}_{(q,p)} \right]$$

$$\stackrel{!}{=} \{f, g\}_{(Q,P)} = \sum_k \left(\frac{\partial f}{\partial Q_k} \frac{\partial g}{\partial P_k} - \frac{\partial g}{\partial Q_k} \frac{\partial f}{\partial P_k} \right)$$

$$\Rightarrow \{Q_k, P_l\} = \delta_{kl} \;, \quad \{Q_k, Q_l\} = 0 \;, \quad \{P_k, P_l\} = 0$$

Note that this also implies - consider 2x2 blocks:

$$(MJM^T)_{jk} = \begin{pmatrix} \{Q_j,Q_k\} & \{Q_j,P_k\} \\ \{P_j,Q_k\} & \{P_j,P_k\} \end{pmatrix} = \begin{pmatrix} 0 & \delta_{jk} \\ -\delta_{jk} & 0 \end{pmatrix}$$

<u>Infinitesimal canonical transformations</u>

$$q_j \to Q_j = q_j + \alpha F_j(q_k, p_k)$$
$$p_j \to P_j = p_j + \alpha E_j(q_k, p_k)$$

What functions are allowed? The (2x2) blocks of the Jacobian are

$$M_{jk} = \begin{pmatrix} \delta_{jk} + \alpha \partial F_j/\partial q_k & \alpha \partial F_j/\partial p_k \\ \alpha \partial E_j/\partial q_k & \delta_{jk} + \alpha \partial E_j/\partial p_k \end{pmatrix}$$

So $MJM^T = J$ implies

$$\sum_k \begin{pmatrix} \delta_{jk} + \alpha \partial F_j/\partial q_k & \alpha \partial F_j/\partial p_k \\ \alpha \frac{\partial E_j}{\partial q_k} & \delta_{jk} + \alpha \partial E_j/\partial p_k \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \delta_{kl} + \alpha \partial F_k/\partial q_l & \alpha \partial E_k/\partial q_l \\ \alpha \frac{\partial F_k}{\partial p_l} & \delta_{kl} + \alpha \partial E_k/\partial p_l \end{pmatrix}$$

$$= \sum_k \begin{pmatrix} \\ \end{pmatrix} \begin{pmatrix} \alpha \frac{\partial F_k}{\partial p_l} & \delta_{kl} + \alpha \frac{\partial E_k}{\partial p_l} \\ -\left(\delta_{kl} + \alpha \frac{\partial F_k}{\partial q_l}\right) & -\alpha \frac{\partial E_k}{\partial q_l} \end{pmatrix}$$

$$= \sum_k \begin{pmatrix} \left(\delta_{jk} + \alpha \frac{\partial F_j}{\partial q_k}\right) \alpha \frac{\partial F_k}{\partial p_l} - \alpha \frac{\partial F_j}{\partial p_k}\left(\delta_{kl} + \alpha \frac{\partial F_k}{\partial q_l}\right) & \left(\delta_{jk} + \alpha \frac{\partial F_j}{\partial q_k}\right)\left(\delta_{kl} + \alpha \frac{\partial E_k}{\partial p_l}\right) - \alpha^2 \frac{\partial F_j}{\partial p_k} \frac{\partial E_k}{\partial q_l} \\ \alpha^2 \frac{\partial E_j}{\partial q_k} \frac{\partial F_k}{\partial p_l} - \left(\delta_{jk} + \alpha \frac{\partial E_j}{\partial p_k}\right)\left(\delta_{kl} + \alpha \frac{\partial F_k}{\partial q_l}\right) & \alpha \frac{\partial E_j}{\partial p_k}\left(\delta_{kl} + \alpha \frac{\partial E_k}{\partial p_l}\right) - \left(\delta_{jk} + \alpha \frac{\partial E_j}{\partial p_k}\right) \alpha \frac{\partial E_k}{\partial q_l} \end{pmatrix}$$

$$= \sum_k \begin{pmatrix} \alpha\left(\delta_{jk} \frac{\partial F_k}{\partial p_l} - \frac{\partial F_j}{\partial p_k} \delta_{kl}\right) & \delta_{jk}\delta_{kl} + \alpha\left(\delta_{jk} \frac{\partial E_k}{\partial p_l} + \frac{\partial F_j}{\partial q_k} \delta_{kl}\right) \\ -\delta_{jk}\delta_{kl} - \alpha\left(\frac{\partial E_j}{\partial p_k} \delta_{kl} + \frac{\partial F_k}{\partial q_l} \delta_{jk}\right) & \alpha\left(\frac{\partial E_j}{\partial p_k} \delta_{kl} - \delta_{jk} \frac{\partial E_k}{\partial q_l}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \alpha\left(\frac{\partial F_j}{\partial p_l} - \frac{\partial F_j}{\partial p_l}\right) & \delta_{jl} + \alpha\left(\frac{\partial E_j}{\partial p_l} + \frac{\partial F_j}{\partial q_l}\right) \\ -\delta_{jl} - \alpha\left(\frac{\partial E_j}{\partial p_l} + \frac{\partial F_j}{\partial q_l}\right) & \alpha\left(\frac{\partial E_j}{\partial q_l} - \frac{\partial E_j}{\partial q_l}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \delta_{jl} \\ -\delta_{jl} & 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 & \left(\frac{\partial E_j}{\partial p_l} + \frac{\partial F_j}{\partial q_l}\right) \\ -\left(\frac{\partial E_j}{\partial p_l} + \frac{\partial F_j}{\partial q_l}\right) & 0 \end{pmatrix} \overset{!}{=} J$$

This means

$$\frac{\partial F_j}{\partial p_\ell} = - \frac{\partial E_j}{\partial p_\ell}$$

which is true if

$$F_j = \frac{\partial G}{\partial p_j} \quad , \quad E_j = - \frac{\partial G}{\partial q_j}$$

for some <u>generator</u> $G(q_k, p_k)$

Suppose we have a one-parameter family of transformations

$$q_j \to Q_j(q_k, p_k ; \alpha) \qquad p_j \to P_j(q_k, p_k ; \alpha)$$

that is an <u>active</u> transformation in phase space, from

$$(q_j, p_j) \quad \longrightarrow \quad (Q_j(q_k, p_k, \alpha), P_j(q_k, p_k ; \alpha))$$

As we vary α, we map out a line in phase space - a <u>flow</u> generated by G. This is evident from the tangents:

$$\frac{dq_j}{d\alpha} = \frac{\partial G}{\partial p_j} \quad , \quad \frac{dp_j}{d\alpha} = - \frac{\partial G}{\partial q_j}$$

$\Rightarrow$ Looks just like ~~the~~ Hamilton's eqns. with $t \to \alpha$, $H \to G$.

Thus, time evolution can be thought of as a canonical transformation <u>generated</u> by <u>the Hamiltonian</u> (analogous to unitary time evolution in QM.)